SPECTRUM-BASED ANALYSIS FOR STABILITY AND ROBUST STABILITY OF DIFFERENTIAL-ALGEBRAIC EQUATIONS

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**Habilitationsschrift an der Fakultät II —
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Dedicated to the memories of my father VU TAM LANG (1933–2004)
and my doctoral supervisor KATALIN BALLA (1947–2005)

PREFACE

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Spectrum-based analysis for stability and robust stability of differential-algebraic equations

(Summary for cumulative *Habilitation*)

Vu Hoang Linh

Contents

1 Introduction

This summary, which is based on nine selected refereed articles [A1-A4, B1-B5] previously published in the period 2005–2011, is concerned with the stability theory of differential-algebraic equations (DAEs). We analyze spectral notions and spectrum-based techniques in the stability analysis of DAEs. Systems of DAEs, which are also called descriptor systems or generalized state-space systems in the control literature, are a very convenient modeling concept in various real-life applications such as mechanical multibody systems, electrical circuit simulation, chemical reactions, semi-discretized partial differential equations, and in general for automatically generated coupled systems, see [8, 24, 27, 35, 39, 45, 46] and the references therein.

DAEs are generalizations of ordinary differential equations (ODEs) in that certain algebraic equations constrain the dynamical behavior. Since the dynamics of DAEs is constrained to a manifold which often is only given implicitly, many theoretical and numerical difficulties arise, which may lead to a sensitive asymptotic behavior of the solution of DAEs to perturbation in the data. The extra difficulties arising with DAEs are characterized by fundamental notions such as regularity, index, and solution subspace, which do not exist for ODEs. Unlike ODEs, the theory of DAEs has a much shorter history. Intensive research in both the theoretical and numerical aspects began in the early 80's, after a remarkable paper of Petzold [44]. In the last two decades several efforts have been made to extend qualitative and quantitative results which are well-known for ODEs to DAEs and this habilitation is focussed in this field. It has turned out that many theoretical results and numerical methods which are valid for ODEs either no longer hold for DAEs or they hold but usually certain extra conditions are required. Extensive lists of references, in particular those concerned with stability issues, can be found in [35, 39, 22] and [B2, B3].

The papers compiled for this summary focus on the stability, the stability robustness, and the spectral analysis of linear DAEs with time-invariant or time-varying coefficients of the form

$$E(t)\dot{x}(t) = A(t)x(t) + f(t), \tag{1}$$

on the half-line $\mathbb{I} = [0, \infty)$, together with an initial condition

$$x(t_0) = x_0, \; t_0 \in \mathbb{I}. \tag{2}$$

Here we assume that $E, A \in C(\mathbb{I}, \mathbb{K}^{n\times n})$, and $f \in C(\mathbb{I}, \mathbb{K}^n)$ are sufficiently smooth. If $E(t)$ is nonsingular for all t, then multiplying both sides of (1) by $E^{-1}(t)$, we arrive at an ODE system. Otherwise, (1) is called a system of DAEs [36]. The coefficient matrices E and A may contain one or several parameters as well. In many control applications, f may depend also on the state x evaluated at one or several delayed times. In this case, we talk about delay DAEs. We use the notation $C(\mathbb{I}, \mathbb{K}^{n\times n})$ to denote the space of continuous functions from $\mathbb{I}$ to $\mathbb{K}^{n\times n}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$.

Linear systems of the form (1) arise directly in many applications and via linearization around solution trajectories [9]. They describe the local behavior in the neighborhood of a solution for general implicit nonlinear system of DAEs

$$F(t, x(t), \dot{x}(t)) = 0, \tag{3}$$

Time-invariant DAEs arise in the case of linearization around stationary solutions.

A function $x : \mathbb{I} \to \mathbb{R}^n$ is called a *solution* of (1) if $x \in C^1(\mathbb{I}, \mathbb{R}^n)$ and x satisfies (1) pointwise. It is called a *solution of the initial value problem* (1)–(2) if x is a solution of (1) and satisfies (2). An initial condition (2) is called *consistent* if the corresponding initial value problem has at least one solution. We note that, by the tractability index approach [26, 39], the condition on the smoothness of solutions may be relaxed, namely, only a part of x is required to be continuously differentiable.

Here, we restrict the discussion to *regular* DAEs, i. e., we require that (1) (or (3) locally) have a unique solution for sufficiently smooth E, A, f (F) and appropriately chosen (consistent) initial conditions. One can immediately extend well-known classical stability concepts for ODEs [1, 12] verbatim to regular DAEs, e.g., see [36]. However, one has to be careful with the initial conditions and the inhomogeneities, since they are restricted due to the algebraic constraints in the system. In other words we require the initial condition (2) be consistent.

The papers discussed in Section 2 are devoted to the stability and the robust stability of two classes of linear time-invariant DAEs. We consider singularly perturbed DAEs, where the leading coefficient depends on a small parameter, and DAEs with delays. For the first class, we present a sufficient condition for the asymptotic stability and investigate the asymptotic behavior of the stability radius, which measures the stability robustness (the distance to instability) of a parameterized DAE system, as the parameter tends to zero. For DAEs with delays, we derive a practically checkable algebraic condition for the asymptotic stability and analyze the stability of stiff integrators. In addition, for a special class of singularly perturbed DAEs with delays, we investigate the asymptotic behavior of the stability radius as the parameter tends to zero.

The papers discussed in Section 3 are devoted to the robust stability and the spectral analysis of linear time-varying DAEs. We obtain a formula for the structured stability radii for time-varying DAEs under dynamic perturbations and investigate the dependence of the stability radii on data. Furthermore, Bohl exponents are shown to be an important tool in the robust stability analysis for DAEs as well. As a major contribution to the stability theory of DAEs, we develop systematically a spectral theory for DAEs. Exponent and spectral notions such as Lyapunov exponents, Bohl exponents, Lyapunov spectrum, and Sacker-Sell spectrum which are well-known for ODEs are extended to DAEs. Moreover, numerical methods based on smooth factorizations like QR and SVD are proposed and analyzed for approximating the spectral intervals and their associated leading directions.

2 Stability and robust stability of linear time-invariant DAEs

Papers to be discussed in this section:

[A1] N.H. Du, V.H. Linh, Implicit-system approach to the robust stability for a class of singularly perturbed linear systems. *Systems Control Letters*, 54: 33–41, 2005.

[A2] V.H. Linh. On the robustness of asymptotic stability for a class of singularly perturbed systems with multiple delays. *Acta Mathematica Vietnamica*, 30:137–151, 2005.

[A3] N.H. Du, V.H. Linh. On the robust stability of implicit linear systems containing a small parameter in the leading term. *IMA Journal on Mathematical Control and Information*, 23:67–74, 2006.

[A4] S. Campbell, V.H. Linh. Stability criteria for differential-algebraic equations with multiple delays and their numerical solutions. *Applied Mathematics and Computation*, 208:397–415, 2009.

Summary

In [A1, A3], we investigate the robust stability of singularly perturbed linear time-invariant DAEs. Roughly speaking, a DAE system is called a singularly perturbed system if it contains a small parameter, usually appearing in the leading term, and setting the parameter to zero causes a change in the algebraic structure of the system, e.g., in the index or in the number of unconstrained equations of the system.

In [A1], we consider the coupled system of implicit linear differential equations

$$\begin{aligned} E_{11}\dot{x}_1(t) &= A_{11}x_1(t) + A_{12}x_2(t) \\ \varepsilon E_{22}\dot{x}_2(t) &= A_{21}x_1(t) + A_{22}x_2(t), \end{aligned} \tag{4}$$

where the solutions $x_i(t)$ are vector functions, $x_i(.) : \mathbb{R} \to \mathbb{C}^{n_i}$, $i = 1, 2$; the coefficients E_{ii}, A_{ij} are constant matrices in $\mathbb{C}^{n_i \times n_i}$ and $\mathbb{C}^{n_i \times n_j}$, respectively, with $i, j = 1, 2$; ε is a small positive parameter. In addition, we assume E_{11} and A_{22} are nonsingular, but E_{22} can be singular. We note that in the case of $\varepsilon = 0$, the system is led to a semi-explicit index-one system of differential-algebraic equations, e.g., see [8]. The system (4) can be considered as a generalization of the classical singular perturbation problem investigated in [20], where E_{ii}, $i = 1, 2$, are set identity matrices. Furthermore, if a singularly perturbed system of second order DAEs that arises, for example, in modeling electrical circuits is converted into a first order system by the standard transformation, then one obtains (4). Assuming that there exists $\varepsilon_0 > 0$ such that (4) is asymptotically stable for all $\varepsilon \in (0, \varepsilon_0]$, we consider the perturbed system

$$\begin{aligned} E_{11}\dot{x}_1 &= (A_{11} + B_1\Delta C_1)x_1 + (A_{12} + B_1\Delta C_2)x_2 \\ \varepsilon E_{22}\dot{x}_2 &= (A_{21} + B_2\Delta C_1)x_1 + (A_{22} + B_2\Delta C_2)x_2 \end{aligned} \tag{5}$$

or in the block form

$$E_\varepsilon \dot{x} = (A + B\Delta C)x \tag{6}$$

with

$$B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, C = \begin{pmatrix} C_1 & C_2 \end{pmatrix}.$$

Here $\Delta \in \mathbb{C}^{p\times q}$ is an uncertain disturbance, B_i, C_i, $i = 1, 2$, are given matrices in $\mathbb{C}^{n_i\times p}$ and $\mathbb{C}^{q\times n_i}$, respectively, and they specify the structure of the perturbation. Following the concept due to Hinrichsen & Pritchard [29, 30], we want to measure the distance to instability for (4) defined by

$$r(E_\varepsilon, A; B, C) = \inf\left\{\|\Delta\|\,,\ \Delta \in \mathbb{C}^{p\times q} \text{ and (6) is not asymptotically stable}\right\} \tag{7}$$

with each $\varepsilon \in (0, \varepsilon_0]$. This quantity is called the *complex structured stability radius* for (4); note that we can also take the algebraic structure preservation into consideration, see [7, 22]. We note that the norm used here is an arbitrary matrix norm induced by a vector norm in $\mathbb{C}^{p\times q}$. When E_{22} is invertible, multiplying both sides of the first and the second equations of (4) or (5) by E_{11}^{-1} and $\varepsilon^{-1}E_{22}^{-1}$, respectively, one obtains an explicit system of ordinary differential equations (ODE-s) which has been well studied in the literature. However, even in this case, one would prefer avoiding the inconvenient computation of the inverse matrices. In addition, because the factor ε^{-1} would appear on the right-hand side of the system, the computation of the stability radius may become an ill-posed problem. Therefore, investigations of the asymptotic stability and the asymptotic behavior of the stability radius $r(E_\varepsilon,\ A;\ B,\ C)$ as ε tends to 0 are of interest. In [20], Dragan considered the problem with identity matrices E_{11}, E_{22}. First, we characterize the asymptotic behavior of the eigenvalues of parameterized pencil $\{E_\varepsilon, A\}$ as ε tends to zero. As a consequence, we obtain a sufficient condition for the asymptotic stability of (4). Namely, if both the "reduced slow system" and the "fast boundary layer system" are asymptotically stable, then there exists $\varepsilon_0 > 0$ such that the singularly perturbed system (4) is asymptotically stable for all $\varepsilon \in (0, \varepsilon_0]$. Then, as a generalization of Dragan's result [20] to (4), but using a completely different approach, we show that when ε tends to zero, the structured stability radius for the singularly perturbed system tends to the smallest value of the stability radius for the "reduced slow system" and that for the "fast boundary layer system". This means that it may happen that $r(E_\varepsilon, A; B, C)$ does not tend to $r(E_0, A; B, C)$ as ε tends to zero. In other words, the complex stability radius as a function of the parameter may be discontinuous. Note that the complex stability radius for an ODE system is well known to depend continuously on data [31].

In [A3], we extend the stability analysis in [A1] to more general parameterized implicit systems of the form

$$(E + \varepsilon F)\dot{x}(t) = Ax(t), \tag{8}$$

where $x(.) : \mathbb{R} \to \mathbb{C}^n$ is a vector function, E, F, A are constant matrices in $\mathbb{C}^{n\times n}$, and ε is a small positive parameter. The leading term $E + \varepsilon F$ may be singular for all sufficiently small ε, but the pencil $\{E+\varepsilon F, A\}$ is assumed to be regular and of index less or equal to one. Here, the matrix F describes the direction of parameter perturbations in the leading term. The first problem is to analyze conditions on the direction F that ensure the asymptotic stability of (8) for all sufficiently small ε. If an appropriate matrix F is chosen, we consider the perturbed system

$$(E + \varepsilon F)\dot{x}(t) = (A + B\Delta C)x(t), \tag{9}$$

and the complex structured stability radius for (8) defined by

$$r(E + \varepsilon F, A; B, C) = \inf \left\{ \|\Delta\|, \Delta \in \mathbb{C}^{p\times q} \text{ and (9) is not asymptotically stable} \right\}. \tag{10}$$

Here once again matrix Δ is an uncertain perturbation and matrices $B \in \mathbb{C}^{n\times p}$, $C \in \mathbb{C}^{q\times n}$ specify the structure of perturbation. The second important question is how the stability radius of an implicit system depends on the leading term. Concretely, under what condition does $r(E + \varepsilon F, A; B, C)$ tend to $r(E, A; B, C)$ as ε tends to zero? What is the asymptotic behavior of $r(E + \varepsilon F, A; B, C)$ for small ε? Since ε is small, the computation of $r(E + \varepsilon F, A; B, C)$ may lead to an ill-posed problem, in general. Therefore, an answer of the latter question is important from both the theoretical and numerical point of view.

In order to answer these questions, first, we classify the parameterized problem (8) into subclasses of singularly perturbed and regularly perturbed systems. In the former case, either the index of pencil or the number of finite eigenvalues of pencil $\{E + \varepsilon F, A\}$ changes as ε becomes zero. In the latter case, the algebraic structure (the index and the number of dynamical equations) is invariant with respect to the parameter. By the approach in [A1], combining with some elementary linear algebra manipulations, we completely characterize the asymptotic behavior of the eigenvalues of the parameterized pencil $\{E + \varepsilon F, A\}$ as ε tends to zero. As a consequence, for singular perturbation problems, if the reduced system and an appropriately constructed "fast boundary layer" subsystem (both may be DAEs) are simultaneously asymptotically stable, then the parameterized system (8) is asymptotically stable for all sufficiently small values of ε. For regular perturbation problems, the asymptotic stability of the reduced system implies the asymptotic stability of the parameterized system (8) for all sufficiently small ε. Next we show that the asymptotic behavior of the complex structured stability radius $r(E + \varepsilon F, A; B, C)$ is quite different in the singular and in the regular perturbation problems. For singular perturbation problems, similar to the result in [A1], $r(E+\varepsilon F, A; B, C)$ converges to the least value of $r(E, A; B, C)$ and the stability radius associated with the "fast" boundary-layer subsystem as ε approaches zero, while in the regular perturbation case, the complex structured stability radius $r(E + \varepsilon F, A; B, C)$ is a continuous function of ε for all sufficiently small (positive and negative) values of ε.

In [A2] and [A4], the asymptotic stability and the robust stability of linear time-invariant DAEs with delays are discussed. In [A2] we consider the singularly perturbed system (SPS) of functional differential equations (FDEs)

$$\begin{aligned} \dot{x}(t) &= L_{11}x_t + L_{12}y_t \\ \varepsilon\dot{y}(t) &= L_{21}x_t + L_{22}y_t \end{aligned} \tag{11}$$

where $x \in \mathbb{C}^{n_1},\ y \in \mathbb{C}^{n_2},\ \varepsilon > 0$ is a small parameter;

$$\begin{aligned} L_{j1}x_t &= \sum_{i=0}^{l} A_{j1}^i x(t - \tau_i) + \int_{-\tau_l}^{0} D_{j1}(\theta)x(t + \theta)d\theta \\ L_{j2}y_t &= \sum_{k=0}^{m} A_{j2}^k y(t - \varepsilon\mu_k) + \int_{-\mu_m}^{0} D_{j2}(\theta)y(t + \varepsilon\theta)d\theta \end{aligned} \tag{12}$$

$j = 1, 2,\ A^i_{jk}$ are constant matrices of appropriate dimensions, $D_{jk}(.)$ are integrable matrix-valued functions, and $0 \le \tau_0 \le \tau_1 \le ... \le \tau_p,\ 0 \le \mu_0 \le \mu_1 \le ... \le \mu_m$.

A number of problems arising in science and engineering can be modeled by SPS-s of differential equations with delay, e.g., see [25] and the references cited therein. Motivated by a stability criterion obtained in [21] and the robust stability analysis in [A1, A3], we first formulate the complex structured stability radii for the parameterized system (11), (12) and for the reduced system (setting $\varepsilon = 0$), which is an index-one semi-explicit DAE system with delays. For the latter system, we require the index to be preserved under perturbation and analyze this carefully by deriving a computable formula for the stability radius. Then, we discuss the asymptotic behavior of the stability radius for the system (11), (12) as the parameter tends to zero. By the idea and framework developed in [A1], we also succeed in proving that the stability radius of the singularly perturbed system converges to the minimum of the stability radii of the reduced DAE system and of a fast boundary-layer system, which is constructed by an appropriate time-scaling.

In [A4], we consider linear differential-algebraic equations with multiple delays

$$A\dot{x}(t) + Bx(t) + \sum_{i=1}^{M} C_i\dot{x}(t-\tau_i) + \sum_{i=1}^{M} D_i x(t-\tau_i) = 0, \tag{13}$$

where $A, B, C_i, D_i, (i = 1, 2, ..., M)$, are real (or complex) constant matrices of size $m \times m$. The time-delays are ordered increasingly, $0 < \tau_1 < \tau_2 < ... < \tau_M$. The matrix A is assumed to be singular with rank$A = d < m$. We also study a special subclass of (13) in the form

$$A\dot{x}(t) + Bx(t) + \sum_{i=1}^{M} C_i\dot{x}(t-i\tau) + \sum_{i=1}^{M} D_i x(t-i\tau) = 0, \tag{14}$$

which is (13) with $\tau_i = i\tau,\ (i = 1, 2, .., M)$, where $\tau > 0$ is given. DAEs with one or several delays in the form (13) appear frequently in control theory, see [48] and the references therein. While the theory and the numerical solution of delay ordinary differential equations (DODEs) is well studied, there are very few results for the theory of delay differential-algebraic equations (DDAEs). The main reason is that even for linear DDAEs, their dynamical behavior is not well-understood yet, in particular when the pair $(A,\ B)$ in (13) is not regular. The most difficult problem is that there exists no compressed form into which a tuple of more than two matrices can be simultaneously transformed. Most of the existing results are only for linear time-invariant regular DDAEs and DDAEs of Hessenberg form. In [A4], first we deal with the solvability of regular DDAEs, i.e., systems with the regular pencil $(A,\ B)$, and DDAEs of Hessenberg form with index up to two. The solvability of some families of non-regular NDDAEs is investigated as well. Then, we discuss spectrum-based stability criteria for regular DDAEs (13). The characteristic equation for (13) is defined by

$$P(s) = \det(sA + B + s\sum_{i=1}^{M} C_i e^{-s\tau_i} + \sum_{i=1}^{M} D_i e^{-s\tau_i}) = 0. \tag{15}$$

For a given $s \in \mathbb{C}$, we denote its real and imaginary parts by $\Re(s)$ and $\Im(s)$, respectively. It is well-known that the system (13) is asymptotically stable if all the roots of (15) have negative real part and they are bounded away from the imaginary axis, i.e., for all root λ_i of (15) $(i = 1, 2, ...)$ and for some positive μ, the inequalities

$$\Re(\lambda_i) \le -\mu < 0 \tag{16}$$

hold. Note that (15) may have infinitely many roots and they may accumulate at a finite point on the complex plane or at infinity. First, by using some results in the perturbation theory of eigenvalue problems [34], we give a delay-independent stability criterion for (13), which improves that given in [50]. However, this sufficient condition is still difficult to check, because the supremum (over the right-half of the complex plane) of the spectral radius of a complicated matrix function has to be calculated which leads to a very complex computational task. Instead, by using a well-known result in the theory of nonnegative matrices and by estimating the differential part and the algebraic part separately, we are able to give rather practical stability criteria, whose efficiency is well illustrated by several numerical examples. These practical stability criteria apply to up-to-index-two NDDAEs (13) (for index-two NDDAEs, certain structure restrictions are needed).

In [A4], we discuss also the asymptotic stability of well-known stiff integrators as they are applied to NDDAEs (13). It is pointed out that under the delay-independent criterion developed in this paper, θ-methods with $\theta \in (1/2, 1]$ and BDF methods are asymptotically stable in the sense that the numerical solutions are asymptotically stable, i.e. these integrators preserve the asymptotic stability of the exact solution. Recently, this result has been extended to strongly A-stable linear multi-step methods [49]. A more general study of stability and robust stability for NDDAEs (13) under structured perturbations, which includes the reduced DDAE discussed in [A2] as a special case, is investigated in [23].

3 Robust stability and spectral theory of linear time-varying DAEs

Papers to be discussed in this section:

[B1] N.H. Du, V.H. Linh. Stability radii for linear time-varying differential algebraic equations with respect to dynamic perturbations. *J. Differential Equations*, 230:579–599, 2006.

[B2] C-J. Chyan, N.H. Du and V.H. Linh. On data-dependence of exponential stability and stability radii for linear time-varying differential-algebraic systems. *J. Differential Equations*, 245:2078–2102, 2008.

[B3] V.H. Linh, V. Mehrmann. Lyapunov, Bohl, and Sacker-Sell spectral intervals for differential-algebraic equations. *J. Dynamics and Differential Equations* 21:153–194, 2009.

[B4] V.H. Linh, V. Mehrmann, E. Van Vleck. QR Methods and Error Analysis for Computing Lyapunov and Sacker-Sell Spectral Intervals for Linear Differential-Algebraic Equations. *Adv. Comput. Math.* 35:281–322, 2011.

[B5] V.H. Linh, V. Mehrmann. Approximation of spectral intervals and associated leading directions for linear differential-algebraic equation via smooth singular value decompositions. *SIAM J. Numer. Anal.* 49:1810–1835, 2011.

Summary

In the papers [B1, B2, B3, B4, B5] listed in this section, we investigate the exponential and asymptotic stability, the robustness, and the qualitative and numerical analysis of Lyapunov and Sacker-Sell spectra for linear time-varying DAEs of the form

$$E(t)\dot{x}(t) = A(t)x(t), \quad t \in \mathbb{I}, \tag{17}$$

with matrix functions $E, A \in C(\mathbb{I}, \mathbb{K}^{n\times n})$, $\mathbb{K} \in \{\mathbb{C}, \mathbb{R}\}$.

Studying the different stability concepts for (17) is more complicated than for linear time-invariant systems. Even if for all $t \in \mathbb{I}$, the finite eigenvalues of $(E(t), A(t))$ have negative real part, system (17) may be unstable, as many well-known examples demonstrate for the ODE case, see e.g., [32].

In the following we assume that (17) is of index at most one, in the sense of the tractability index [26] or that it is strangeness-free in the sense of the strangeness-index [35]. These conditions are equivalent if the coefficients are sufficiently smooth [38]. In the tractability index approach, we assume that $S := \ker E$ is absolutely continuous. Let Q be an absolutely continuous projector onto S, and define $P = I - Q$, a projector along S, and $G := E - (A + E\dot{P})Q$. Then, the DAE system (17) is decomposed into a coupled system consisting of a differential system and an algebraic one, and the solvability as well as the construction of the fundamental matrix become transparent.

The concept of the strangeness index uses the DAE and its derivatives to construct a so-called strangeness-free DAE with the same solution [35]. In the homogenous case this *strangeness-free* system has the form (17), where

$$E = \begin{bmatrix} E_1 \\ 0 \end{bmatrix}, \ A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}, \tag{18}$$

with $E_1 \in C(\mathbb{I}, \mathbb{K}^{d\times n})$ and $A_2 \in C(\mathbb{I}, \mathbb{K}^{(n-d)\times n})$ such that the matrix $\hat{E}(t) := \begin{bmatrix} E_1(t) \\ A_2(t) \end{bmatrix}$ is invertible for all $t \in \mathbb{I}$.

By using a global kinematic equivalence transformation, see [B3, Remark 13], (17) can be further transformed into the special form

$$\begin{bmatrix} \tilde{E}_{11} & \tilde{E}_{12} \\ 0 & 0 \end{bmatrix} \dot{\tilde{x}} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{bmatrix} \tilde{x}, \tag{19}$$

so that an *essential underlying* (implicit) ODE system is readily given by

$$\tilde{E}_{11}\dot{x}_1 = \tilde{A}_{11}x_1 \tag{20}$$

with nonsingular $\tilde{E}_{11}$.

In [B1, B2] we propose a formula for the stability radius of (17) and analyze its dependence on data. While the formula of the complex stability radius for linear time-invariant systems can be obtained in a straightforward way, that for linear time-varying systems is more difficult [28] and it had been an open problem until Jacob's paper [33], where the problem was solved using the theory of causal operators.

In [B1], we extend Jacob's result to linear time-varying DAEs (17). We assume that (17) is robustly index-one [B1, Assumption A2] or [22, Definition 3.4], exponentially stable, and subject to structured perturbation of the form

$$E(t)\dot{x}(t) = A(t)x(t) + B(t)\Delta(C(\cdot)x(\cdot))(t), \quad t \in \mathbb{I}, \tag{21}$$

where $B \in L_\infty(0,\infty;\mathbb{K}^{n\times m})$ and $C \in L_\infty(0,\infty;\mathbb{K}^{q\times n})$ are given matrix functions, restricting the structure of the perturbation and $\Delta : L_p(0,\infty;\mathbb{K}^q) \to L_p(0,\infty;\mathbb{K}^m)$ is an unknown causal perturbation operator. Such a perturbation is called dynamic since the perturbation depends dynamically on the state x. In general, IVPs for (21) do not have solutions in the classical sense, instead we consider mild solutions [B1, Definition 3]. Furthermore, we require the mild solutions of the IVPs for (21) be globally L_p-stable for $1 \le p < \infty$, see [B1, Definition 5]. We associate with the perturbed system (21) the perturbation operator (in [33, A3] it is called the input-output operator)

$$\mathbb{L}_{t_0} z = C\int_{t_0}^{t} \Phi(t,\rho)PG^{-1}B(\rho)z(\rho)d\rho + CQG^{-1}Bz, \tag{22}$$

which is defined for all $t \ge t_0 \ge 0$, $z \in L_p(0,\infty;\mathbb{K}^m)$. It is important to note that the mild solution as well as the perturbation operator are independent of the choice of projector Q. Furthermore, this operator is linear, bounded, and monotonically non-increasing with respect to t.

The *complex/real structured stability radii* of (17) subject to dynamic structured perturbation as in (21), are defined by

$$r_{\mathbb{K}}(E,A;B,C) = \inf\left\{ \begin{array}{r} \|\Delta\|\,, \text{ the trivial solution of (21) is not globally } L_p-\text{stable} \\ \text{or (21) is not of index one} \end{array} \right\},$$

where $\mathbb{K}=\mathbb{C}$ or $\mathbb{K}=\mathbb{R}$, respectively. As the main result of [B1], the following formula for the stability radii is proven

$$r_{\mathbb{K}}(E,A;B,C) = \min\{\lim_{t_0\to\infty} \|\mathbb{L}_{t_0}\|^{-1}, (\text{ess sup}_{\ t\in\mathbb{I}} \left\|CQG^{-1}B(t)\right\|)^{-1}\}. \tag{23}$$

Comparing with the proof in the ODE case [33], we have to overcome more sophisticated manipulations and in particular, to pay a careful attention to the preservation of the index-one property which is an extra difficulty. If all the coefficients are real, then the complex stability radius and the real one are equal. In addition, as a special case, we show that for linear time-invariant DAEs and $p = 2$, the complex structured stability radii with respect to dynamic and static perturbations (that is the case when B, C, and Δ are constant matrices) coincide.

In [B2], we investigate how the stability radii $r_{\mathbb{K}}(E,A;B,C)$ depend on the data set $\{A,B,C\}$. We consider the perturbed system

$$E(t)\dot{x}(t) = (A(t)+H(t))x(t), \quad t \in \mathbb{I}, \tag{24}$$

with a perturbation function $H \in L_\infty(0,\infty;\mathbb{K}^{n\times n})$. First, we show that if the uncertain perturbation H is sufficiently small in a certain sense, then the perturbed system (24) remains index one and exponentially stable. To this end, we use the concept of Bohl exponents [6, 12].

The *Bohl exponent* for an index-one system of the form (17)

$$\begin{aligned} k_B(E,A) \;&=\; \inf\{-\alpha \in \mathbb{R};\ \text{there exists } \ L_\alpha > 0 \\ &\qquad \text{such that for all } t \ge t_0 \ge 0 : \|\Phi(t,t_0)\| \le L_\alpha e^{-\alpha(t-t_0)}\Big\}, \end{aligned}$$

where Φ is the fundamental matrix of the DAE. It follows that (17) is exponentially stable if and only if its Bohl exponent is negative (including the case $k_B(E,A) = -\infty$). Furthermore, the Bohl exponent of (17) can be determined by

$$k_B(E,A) = \limsup_{s,t-s\to\infty} \frac{\ln \|\Phi(t,s)\|}{t-s}.$$

In [B2], properties of the Bohl exponent, as well as the connection between the exponential stability of (17) and the boundedness of solutions to nonhomogeneous DAE with bounded inhomogeneity are investigated. We show that the Bohl exponent for (17) is stable to perturbations H in the following sense. Suppose that H satisfies

$$\sup_{t\in\mathbb{I}} \|H(t)\| < (\sup_{t\in\mathbb{I}} \|QG^{-1}(t)\|)^{-1}. \tag{25}$$

and, in addition, that for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that

$$\limsup_{s,t-s\to\infty} \frac{1}{t-s}\int_s^t \|PG^{-1}H(\tau)\|d\tau < \delta, \tag{26}$$

then

$$k_B(E,A+H) < k_B(E,A) + \varepsilon.$$

Consequently, if the supremum norm of H on $\mathbb{I}$ tends to zero, then the Bohl exponent of the perturbed system (24) converges to that of the unperturbed system (17), i.e., the Bohl exponent of (17) depends continuously on the coefficient A. Thus, if (17) is robustly index one and exponentially stable, then (24) with H satisfying (25),(26) is exponentially stable, too.

As main result of [B2], we characterize the sequence of perturbations H_k with the supremum norm tending to zero such that the stability radius of the perturbed systems with the same perturbation structure converges to that of the unperturbed system. As a consequence to the numerical approximation of the stability radius, the stability radius of an asymptotically constant-coefficient system of the form (24) can be well approximated by that of linear time-invariant system for which efficient numerical methods exist, e.g., see [3, 7]. It is of practical importance, because the computation of the stability radius for linear time-varying system (17) via the formula (23) seems to be practically impossible. In [B2], we also show that the stability radius $r_{\mathbb{K}}(E,A;B,C)$ depends continuously on the perturbation-structure prescribed by matrix functions B and C. Recently, some of the results of [B1, B2] have been further extended in a similar framework [4, 5].

The spectral theory for ODEs has a long history with important work by Lyapunov, Perron, Bohl, etc, see [1, 12, 47] and the references therein. In the last decade, the numerical computation of spectral intervals based on smooth factorizations QR and SVD has become feasible and well analyzed for ODEs, mainly due to a number of works by Dieci and Van Vleck, [13, 14, 15, 16, 17, 18]. The spectral theory and numerical methods for the approximation of spectral intervals in the case of DAEs, however, are still in their infancy. Starting just recently with [B3] and continuing with a sequence of works [B4, B5, 40, 41, 42], we have extended the spectral theory and the numerical methods for approximating spectral intervals and associated leading directions to DAEs. Here we summarize the main results of [B3, B4, B5]. Instead of the projector-based index approach used in [B1, B2], we apply the strangeness-index approach and consider (17) in the strangeness-free form (18) because of the numerical feasibility of the latter index approach.

In [B3], by using orthogonal changes of variables, the original DAE system is transformed into appropriate condensed forms, for which concepts such as Lyapunov exponents, Bohl exponents, exponential dichotomy, and spectral intervals of various kinds can be analyzed via the resulting underlying ODEs. Furthermore, it is often convenient to transform (18) into other forms which are easier to handle but their solutions have the same asymptotic behavior as those of the original system. Suppose that $P \in C(\mathbb{I}, \mathbb{R}^{n\times n})$ and $Q \in C^1(\mathbb{I}, \mathbb{R}^{n\times n})$ are nonsingular matrix functions such that Q and Q^{-1} are bounded. Then the transformed DAE system

$$\tilde{E}(t)\dot{\tilde{x}} = \tilde{A}(t)\tilde{x}, \tag{27}$$

with $\tilde{E} = PEQ$, $\tilde{A} = PAQ - PE\dot{Q}$ and $x = Q\tilde{x}$ is called *globally kinematically equivalent* to (18) and the transformation is called a *global kinematic equivalence transformation.* If $P \in C^1(\mathbb{I}, \mathbb{R}^{n\times n})$ and, furthermore, also P and P^{-1} are bounded then we call this a *strong global kinematic equivalence transformation.* This is the key idea for the theory and the numerical methods developed in [B3].

For a given fundamental solution matrix X of a strangeness-free DAE system of the form (18), and for $d \le k \le n$, we introduce

$$\lambda_i^u = \limsup_{t\to\infty} \frac{1}{t} \ln \|X(t)e_i\| \text{ and } \lambda_i^\ell = \liminf_{t\to\infty} \frac{1}{t} \ln \|X(t)e_i\|, \quad i = 1, 2, ..., k,$$

where e_i denotes the i-th unit vector. The columns of a minimal fundamental solution matrix form a *normal basis* if $\Sigma_{i=1}^d \lambda_i^u$ is minimal. The $\lambda_i^u, i = 1, 2, ..., d$, belonging to a normal basis are called *(upper) Lyapunov exponents* and the intervals $[\lambda_i^\ell, \lambda_i^u], i = 1, 2, ..., d$, are called *Lyapunov spectral intervals.* The set of the Lyapunov spectral intervals is called the *Lyapunov spectrum* of (18).

We show that by an orthogonal transformation, the so-called *essential underlying ODE system* (EUODE) can be obtained, which possesses the same Lyapunov spectra as (18). Applying an orthogonal change of basis, we transform the system (18) into the condensed form (19) or alternatively into

$$\begin{bmatrix} \hat{E}_{11} & 0 \\ 0 & 0 \end{bmatrix} \dot{\hat{x}} = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix} \hat{x}, \quad t \in \mathbb{I}, \tag{28}$$

from which essential underlying ODE systems are easily obtained.

In parallel with (18), we consider the associated adjoint DAE [10, 2, 37], which is defined by

$$\frac{d}{dt}(E^T y) = -A^T y, \quad \text{ or } \quad E^T(t)\dot{y} = -[A^T(t) + \dot{E}^T(t)]y, \quad t \in \mathbb{I}. \tag{29}$$

Then, under certain boundedness conditions for the coefficients, we show that the Lyapunov spectrum of (18), that of the EUODEs, and that of the adjoint DAE (29) coincide. As a consequence, (18) is Lyapunov-regular, see [B3, Definition 17], if and only if its EUODEs (or its adjoint DAE) are so. Several examples are given to illustrate the differences between the Lyapunov theory of ODEs and that of DAEs.

Another important question, namely the stability of Lyapunov exponents, is investigated, as well. Roughly speaking, we ask whether a small perturbation in the coefficients induces only small changes in the Lyapunov exponents. For ODEs, it is well-known that the existence of an integrally separated fundamental solution matrix, see [B3, Definition 30] and [1, Chapter 5], is a necessary and sufficient condition for the stability of distinct Lyapunov exponents. We show that in addition to the integral separation, the robustness of strangeness-free property and some extra boundedness conditions are required to ensure the stability of the Lyapunov exponents. This means that the Lyapunov exponents of even a strangeness-free DAE system are more sensitive to perturbations than those of an ODE system (even when the algebraic structure of DAEs is preserved), see [40, Example 4.2] for illustration. Furthermore, it is shown that the integral separation holds simultaneously for (18) and its adjoint (29).

In addition to the Lyapunov exponents and the Lyapunov spectrum, other notions, namely the Bohl exponents and the Sacker-Sell spectrum, play very important roles in the stability analysis of dynamical systems described by ODEs [6, 12, 47]. As we discussed in the summary of [B2], the Bohl exponent is proven to be useful in the robust stability analysis. In [B3], we define the upper and lower Bohl exponents for any nontrivial solution of (18) and the Bohl spectral intervals for a given minimal fundamental solution matrix. While the Lyapunov exponents characterize the asymptotic/expoential growth rate of a solution, the Bohl exponents measure the *uniform* exponential growth rate. The relation between the Bohl intervals for (18) and those for the corresponding EUODE as well as the properties of the Bohl intervals of an integrally separated fundamental solution matrix are analyzed.

In order to extend the concept of exponential dichotomy (Sacker-Sell) spectrum to DAEs, we introduce the shifted DAEs for (18), see [B3, Definition 41], and the concept of exponential dichotomy, see [B3, Definition 44]. For sake of simplicity, we restrict our investigation to the transformed DAE system (28). This restriction is relaxed in [B5]. It is shown that the exponential dichotomic property hold simultaneously for (28) and its EUODE. Furthermore, this property is invariant under global kinematic equivalence transformations. The *Sacker-Sell spectrum* of (28) is defined by the set of all real shift parameters with which the corresponding shifted DAEs do not have an exponential dichotomy. Then, comparisons of the Sacker-Sell spectrum with the Lyapunov spectrum and with the set of Bohl spectral intervals are given. An important auxiliary result concerning the transformation of an implicit ODE system into upper triangular form is stated, from which the equivalence between the Sacker-Sell spectrum of (28) and that of its adjoint becomes clear.

For ODEs, it is well-known that the Sacker-Sell spectrum is stable to perturbations. In [B3], the stability of the Sacker-Sell spectrum for (28) is investigated. By invoking the Roughness Theorem [11], explicit estimates for the movement of the Sacker-Sell spectral intervals under small perturbations occurring in the coefficients are presented.

Numerical methods based on QR decompositions are proposed in [B3]. The idea is first to transform (17) to (19) in order to obtain the EUODE (20), which is an implicit ODE system of reduced size. Then, similarly to the ODE case, discrete and continuous QR methods are presented. In the discrete QR method, the fundamental matrix of (18) is integrated by a DAE solver in subsequent intervals with reorthogonalization applied at the beginning of each interval. Then, QR factorizations are performed for the fundamental matrices on the subintervals and the diagonal elements of triangular factors are used to approximate the endpoint of spectral intervals. In the continuous QR method, we derive the differential equations for the orthogonal factor Q and for the diagonal elements of the upper triangular factor R. Then Q and the diagonal of R are integrated by appropriate solvers. As an example, we use the simple Euler method combined with projection since Q must be kept orthogonal at the mesh-points. Numerical examples are given which illustrate and confirm the efficiency of these QR methods. We note that to use the continuous QR method, one must implement very carefully the orthogonal pre-transformation that brings (18) into the form (19).

In [B4, B5], the spectral theory and the numerical methods initiated in [B3] are further developed and improved. In [B4], first the numerical methods based on QR factorizations proposed in [B3] are improved. The main difference is that now the methods are applied directly to (18). In the continuous fashion, the equation for factor Q is a semi-linear matrix DAE system instead of an ODE system as in [B3]. We also give a comparison of this approach with the previous one in [B3]. Then, as the main result, we present a rigorous perturbation and error analysis which verifies the applicability of these methods. As a matter of fact, we show here that, although we need to numerically integrate some DAE systems on usually very long time-intervals, the error in the computed spectral intervals depends essentially only on the local error of the numerical integration, the error arising in the solution of the algebraic constraint equations, and on the degree to which the DAE is integrally separated. These errors, however, can be easily kept under control by using an appropriate integration method for strangeness-free DAEs accompanied with a local error estimator and stepsize control, while integral separation is a natural and prevalent structural condition that is also central to the robustness of Lyapunov exponents. Our emphasis in this work is on strangeness-free DAEs that enjoy the integral separation property.

Applying a similar framework as in [17, 18, 19], a backward and forward error analysis is carried out. In the backward error analysis, we show that the numerically computed QR factorization of the fundamental matrix is the exact QR factorization of the fundamental matrix of a nearby perturbed problem and the perturbation in norm has the same magnitude as the local integration error. Here, for the discrete QR method we require the numerical solution of (18) satisfy exactly the algebraic equation and for the continuous QR method the computed Q factor must satisfy the orthogonality condition at the mesh-points. These requirements are to be taken carefully in the implementation of the methods. In the forward error analysis, we propose perturbation bounds for the coefficients of the orthogonal transformation that brings a perturbed upper triangular implicit system into upper triangular form. Combining both error analyses, we are able to give explicit error bounds for the computed Lyapunov and Sacker-Sell spectral intervals. The obtained error bounds depend essentially on the local integration error when solving the DAE system (18) (the discrete QR) or that for factor Q (the continuous QR) and on the conditioning of the orginal DAE system. Numerical experiments confirm the theoretical results.

In [B5], we complete the spectral analysis for (18) by the characterization of leading directions and subspaces associated with spectral intervals, which are generalizations of eigenvectors and invariant subspaces in the time-invariant setting. With respect to the fact that (18) and its EUODE possess the same spectral properties, this is a straightforward extension from ODEs to DAEs. Next, using the approach of [B4], we discuss the extension of the methods based on smooth singular value decompositions (SVDs) introduced in [13, 14] to DAEs. In fact, the construction of EUODEs by using different bases of the solution subspace provides a unified insight into different techniques such as the QR- and SVD-based methods for the approximation of spectral intervals for DAEs. Under the integral separation condition, these SVD based methods apply directly to DAEs (18) to compute the spectral intervals and their associated leading directions. Most of the theoretical results as well as the numerical methods are direct generalizations of [13] but, in addition, we prove that the limit (as t tends to infinity) of the V-component in the smooth SVD of any fundamental solution provides not only a normal basis, but also an integrally separated fundamental solution matrix, see [B5, Theorem 4.11]. This significantly improves Theorem 5.14 and Corollary 5.15 in [13].

We also give a detailed discussion on the comparison of the continuous QR and SVD based methods in [B5]. It turns out that the SVD based algorithms require a more careful implementation in order to avoid numerical instability, but we do not need to start with a normal basis as in the case of the QR based methods. We can choose any fundamental matrix and proceed with it. Further, the integration of factor V can be incorporated into the algorithm to obtain the leading directions associated with the spectral intervals. It is shown that the factor V converges exponentially to a constant matrix (whose columns give the leading directions) and the convergence rate depends on the gaps between the exponents. The numerical integration of the orthogonal factor U needs a particular attention, because both the orthogonality and the algebraic constraint must be kept at mesh-points. The use of so-called half-explicit integrators, combined with post-projection, to the well-known implicit DAE integrators is also discussed. The analysis of half-explicit methods that are based on explicit one-leg, linear multi step, and Runge-Kutta methods is given in a separate work [41].

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